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A Remark on Circulants

H. B. Mann

G. Marsaglia

Mathematics Research

August 1962

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A REMARK ON CIRCULANTS

by

H. B. Mann* and G. Marsaglia

* Ohio State University

(Visiting Staff Member)
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Mathematics Research Laboratory

BOEING SCIENTIFIC RESEARCH LABORATORIES

August 1962

Let S be a circulant matrix:

$$S = \begin{pmatrix} a_0 & a_1 & a_2 & \cdots & a_{n-1} \\ a_{n-1} & a_0 & a_1 & \cdots & a_{n-2} \\ a_{n-2} & a_{n-1} & a_0 & \cdots & a_{n-3} \\ \vdots & & & & \\ a_1 & a_2 & a_3 & \cdots & a_0 \end{pmatrix}$$

that is, a matrix in which each row is obtained from the previous one by shifting elements one position to the right and bringing the last element to the first position. Such arrays are also called cyclic matrices. The determinants of such matrices are called circulants, and have been extensively studied, see for example, Muir [1], [2], or [3], [4] for more recent mention. References [5] and [6] give examples of circulants arising in physical theories.

If only determinants are considered, it doesn't matter whether the cyclic permutation that forms the successive rows of S is a shift of one to the right or a shift of one to the left, but the properties of the matrices so formed are very easily established for the 1-shift right and not so easily for the 1-shift left, even though the latter are symmetric.

Generalizations of S in which successive rows are formed by a k-shift right have been considered, as well as generalizations in which the elements of S are themselves matrices, [4], [7].

We wish to point out that the classical results, as well as some of the more recent results, can be established in a very simple manner. The method is completely general and includes the case of fields with finite characteristic. Circulants with elements from such fields are useful in certain combinatorial problems.

Let S be the matrix above, formed by a 1-shift right, and let P be the permutation matrix that effects that shift:

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ 1 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

Then $P^n = I$, and in fact the minimum polynomial of P is $x^n - 1$, for it is easy to verify that

$$(1) \qquad a_0 I + a_1 P + a_2 P^2 + \cdots + a_{n-1} P^{n-1} = S$$

is the circulant matrix given above, and $S = 0$ only when the a's vanish.

Thus, if

$$f(x) = a_0 + a_1 x + \cdots + a_{n-1} x^{n-1}$$

then $S = f(P)$, and the characteristic roots of S are

$$f(r_1), f(r_2), \ldots, f(r_n)$$

where $r_1, r_2, \ldots, r_n$ are the characteristic roots of P, that is, the constants in the factorization

$$(2) \qquad x^n - 1 = (x - r_1)(x - r_2) \cdots (x - r_n)$$

of $x^n - 1$ in its splitting field.

The roots of unity will be distinct unless the field has characteristic p that divides n:

$$n = p^t m, \qquad t \geq 1, \ (p,m) = 1.$$

In that case, there will be m distinct roots of unity, each of multiplicity p^t, say, $r_1, r_2, \ldots, r_m$.

Then

$$x^n - 1 = (x^m - 1)^{p^t} = (x - r_1)^{p^t}(x - r_2)^{p^t} \cdots (x - r_m)^{p^t}$$

and the $n = p^t m$ characteristic roots of S can be listed in the form:

$$\begin{array}{l} f(r_1), f(r_1), \ldots, f(r_1) \\ f(r_2), f(r_2), \ldots, f(r_2) \\ \cdot \\ \cdot \\ \cdot \\ f(r_m), f(r_m), \ldots, f(r_m) \end{array}$$

with each row having p^t entries.

The product of the characteristic roots of S gives its determinant, and hence most of the known results on circulants - see particularly [2], vol. II, pp. 401-412, vol. III, pp. 372-392, or [3], section 7, [4] Theorem 2, [7] Theorem 3, for results on fields of characteristic p.

We can establish more properties of S by studying P. Again with $r_1, r_2, \ldots, r_n$ given by the factorization of $x^n - 1$ over its splitting field in (2), let V be the Vandermonde matrix

$$V = \begin{pmatrix} 1 & r_1 & r_1^2 & \cdots & r^{n-1} \\ 1 & r_2 & r_2^2 & \cdots & r_2^{n-1} \\ \cdot & & & & \\ \cdot & & & & \\ \cdot & & & & \\ 1 & r_n & r_n^2 & \cdots & r_n^{n-1} \end{pmatrix}$$

Straightforward multiplication will verify that

$$VP = DV$$

where D is diagonal:

$$D = \begin{pmatrix} r_1^{-1} & 0 & \cdots & 0 \\ 0 & r_2^{-1} & \cdots & 0 \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ 0 & 0 & \cdots & r_n^{-1} \end{pmatrix}$$

If $r_1, \ldots, r_n$ are distinct, then V has an inverse, in fact $V^{-1} = n^{-1}\bar{V}'$, so that $U = n^{-1/2}V$ is unitary, $U\bar{U}' = I$.

Thus if p does not divide n, the matrix V diagonalizes P and all circulants S simultaneously:

$$VSV^{-1} = \begin{pmatrix} f(r_1^{-1}) & 0 & \cdots & 0 \\ 0 & f(r_2^{-1}) & \cdots & 0 \\ 0 & 0 & \cdots & f(r_n^{-1}) \end{pmatrix}$$

If p divides n then the minimum polynomial $x^n - 1$ of P has multiple roots and P cannot be diagonalized; the question of whether S is similar to a diagonal matrix must then be resolved for each S separately.

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